Solutions to Discussion Problems for Math 180

Thursday, March 5, 2015

Review

1. If $f(x) = \tan^{-1}(3x-2)$, what is $f^{-1}(x)$?

$$x = \tan^{-1}(3y-2) \quad \Rightarrow \quad \tan(x) = 3y-2 \quad \Rightarrow \quad y = \frac{\tan(x)+2}{3}$$

2. Find the first and second derivatives of each function:

   (a) $5\sin(x) - 4\cos(x)$

$$f'(x) = 5\cos(x) + 4\sin(x); \qquad f''(x) = -5\sin(x) + 4\cos(x)$$

   (b) xe^{-x^2}

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1-2x^2); \qquad f''(x) = e^{-x^2}(-2x)(1-2x^2) + e^{-x^2}(-4x)$$

   (c) $\tan^{-1}(x)$

$$f'(x) = \frac{1}{1+x^2}; \qquad f''(x) = \frac{-1}{(1+x^2)^2}(2x)$$

   (d) $\dfrac{2x-3}{x-5}$

$$f'(x) = \frac{(x-5)(2)-(2x-3)}{(x-5)^2} = \frac{-7}{(x-5)^2}; \qquad f''(x) = \frac{14}{(x-5)^3}$$

3. For what positive value of x is x^x the smallest?

First we need the derivative of x^x. If $y = x^x$ then

$$\ln(y) = \ln(x^x) = x\ln(x),$$

so, taking the derivative of each side with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + x\frac{1}{x} = \ln(x) + 1.$$

Solving,

$$\frac{dy}{dx} = y(\ln(x) + 1) = x^x(\ln(x) + 1).$$

At a minimum, the derivative vanishes, so to find derivatives we should check when

$$x^x(\ln(x) + 1) = 0.$$

Since x^x is positive for $x > 0$, it cannot be zero, so the only way this equation could hold would be if

$$\ln(x) + 1 = 0 \quad \Rightarrow \quad \ln(x) = -1 \quad \Rightarrow \quad x = e^{-1} = \frac{1}{e}$$

and then it remains only to check that this is indeed a minimum, which we can do e.g. by noting that the derivative is negative for $x < 1/e$ and positive for $x > 1/e$.

4. Prove that $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$ using implicit differentiation.

We want the derivative of $y = f^{-1}(x)$. Rewriting this as $f(y) = x$, we can take derivatives of each side with respect to x, giving

$$f'(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

This time

5. On which intervals is xe^{-x^2} increasing? Decreasing?

From 2(b), the derivative is $e^{-x^2}(1 - 2x^2)$, so our critical points are given by

$$e^{-x^2}(1 - 2x^2) = 0.$$

Since e^{-x^2} must be positive, it cannot be zero, so the only way to satisfy this equation is for

$$1 - 2x^2 = 0 \quad \Rightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x = \pm\frac{\sqrt{2}}{2}.$$

Checking signs, we see that this function is increasing on $(-\sqrt{2}/2, \sqrt{2}/2)$ and decreasing elsewhere.

6. On which intervals are the following functions concave up? Concave down?

   (a) $x^4 - 2x^3 + 1$

   $$f'(x) = 4x^3 - 6x^2$$

   $$f''(x) = 12x^2 - 12x = 12x(x-1)$$

   So the potential inflection points are at $x = 0$ and $x = 1$. Checking signs, we see that f is concave up on $(-\infty, 0) \cup (1, \infty)$ and concave down on $(0, 1)$.

   (b) $\dfrac{2x-3}{x-5}$

   From 2(d),

   $$f''(x) = \frac{14}{(x-5)^3},$$

   which is positive for $x > 5$ and negative for $x < 5$. It follows that f is concave down on $(-\infty, 5)$ and concave up on $(5, \infty)$.

7. Sketch the graph of a differentiable function $f(x)$ on $(-\infty, 0) \cup (0, \infty)$ such that $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $x > 0$, and $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 2$.

   Many possible answers.

8. A patient comes into the emergency room with a myocardial infarction. You administer nitroglycerin as a vasodilator, causing the radii of the blood vessels to increase by 2% per minute. The Hagen-Poiseuille equation from fluid dynamics tells us that the blood flow through a vessel is directly proportional to the fourth power of its radius. The flow must increase by at least 10% per minute or your patient will die. What happens?

   We're told $Q = kr^4$. We want to know about Q'/Q and r'/r, so one thing we can do is notice that $\ln(Q) = \ln(k) + 4\ln(r)$. Taking derivatives with respect to time, we have

   $$\frac{1}{Q}\frac{dQ}{dt} = 4\frac{1}{r}\frac{dr}{dt} = 4(2\%/\text{min}) = 8\%/\text{min}.$$

   This is less than 10%/min, so you will have to do something other than simply administering the nitroglycerin or your patient will die. *DISCLAIMER: I am not a physician, these numbers are completely made up, and this does not constitute any kind of medical advice.*